9/16/09

20. Let I be the $n \times n$ matrix whose entry in row i and column j is

$$\begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Show that $AI = IA = A$ for every $n \times n$ matrix A.

$n=3$

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

identity matrix.

Start w/ $A = [a_{ij}]_{m \times n}$

$$(A * I)_{ij} = (i^{th} \text{ row of } A) \cdot (j^{th} \text{ column of } I)$$

$$= \begin{bmatrix} a_{i1} & a_{i2} & a_{i3} & \dots & a_{in} \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow j^{th} \text{ spot}$$

$$= a_{ij}$$

$$= (A)_{ij}$$

Finish 1.3 from yesterday:

Matrix multiplication: column view

A is m x n, B is n x s, C = AB is m x s

Each column of AB is formed as A*(column of B):

$$AB = A \begin{bmatrix} \vec{b}_1 \mid \vec{b}_2 \mid \dots \mid \vec{b}_s \end{bmatrix} = \begin{bmatrix} A\vec{b}_1 \mid A\vec{b}_2 \mid \dots \mid A\vec{b}_s \end{bmatrix}$$

Yesterday we saw that:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 & 1 \\ 0 & 3 & 5 & 2 \\ 2 & 4 & 10 & -2 \end{bmatrix} = \begin{bmatrix} 7 & 17 & 40 & -1 \\ 16 & 35 & 85 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 & 1 \\ 0 & 3 & 5 & 2 \\ 2 & 4 & 10 & -2 \end{bmatrix} = \begin{bmatrix} 7 & 17 & 40 & -1 \\ 16 & 35 & 85 & 2 \end{bmatrix}$$

as columns instead

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 & 1 \\ 0 & 3 & 5 & 2 \\ 2 & 4 & 10 & -2 \end{bmatrix} = \left[\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \middle| \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} -1 \\ 3 \\ 4 \end{bmatrix} \middle| \cdots \right]$$

$$= \begin{bmatrix} 7 & 17 & \cdots \\ 16 & 35 & \end{bmatrix}$$

Write this out generically in terms of indices:
If A is m x n , B is n x s, then C = AB will be m x s with the
c_ij entry given by ….

$$A = [a_{ij}]_{m \times n}, \quad B = [b_{ij}]_{n \times s}$$

$$C = [c_{ij}]_{m \times s} \rightarrow c_{ij} = (\text{row } i \text{ of } A) \cdot (\text{col } j \text{ of } B)$$

$$= \begin{bmatrix} a_{i1} & a_{i2} & \cdots & a_{in} \end{bmatrix} \cdot \begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{bmatrix}$$

$$= a_{i1} b_{1j} + a_{i2} b_{2j} + \ldots a_{in} b_{nj}$$

$$\boxed{c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}}$$

Matrix multiplication: row view

A is m x n, B is n x s, C = AB will be m x s
Each **row** of AB is formed as (row of A)*B

$$AB = \begin{bmatrix} a_1 \\ \hline a_2 \\ \hline \vdots \\ \hline a_m \end{bmatrix} B = \begin{bmatrix} a_1 B \\ \hline a_2 B \\ \hline \vdots \\ \hline a_m B \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 & 1 \\ 0 & 3 & 5 & 2 \\ 2 & 4 & 10 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 & 1 \\ 0 & 3 & 5 & 2 \\ 2 & 4 & 10 & -2 \end{bmatrix} \\ \hline \begin{bmatrix} 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} & \text{"} & \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 7 & 17 & \cdots \\ & \cdots & \end{bmatrix}$$

~~Can always paint a column picture, each column of AB is composed of columns linear combinations of the columns of A with the coefficients specified in the columns of B (see Ex. 8 or 9).~~

Matrix Transpose: = switching the rows & columns $(i \leftrightarrow j)$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \rightarrow A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

$$(A^T)_{ij} = (A)_{ji}$$

Trace of a square matrix: A is $n \times n$

$$tr(A) = \sum_{i=1}^{n} a_{ii} = a_{11} + a_{22} + \cdots a_{nn}$$

eg $A = \begin{bmatrix} 5 & 9 \\ 3 & -2 \end{bmatrix} \Rightarrow tr(A) = 5 + (-2) = 3$

1.4 Inverses & rules of matrix arithmetic
HW: 1b, 2b, 4, 6, 7bd, 8, 9a, 15-17, 29, 31, 34

Matrix addition is *commutative*: A+B=B+A (so is subtraction),
but the same does not need to be true for multiplication - why?

For one thing even if AB is defined, BA need not be, why?

Try a random example with 2x2 matrices.

Many familiar arithmetic rules work - as long they don't involve commuting matrix multiplication

See Theorem 1.4 in book

Let's prove (e).

Not only is matrix multiplication, in general, not commutative, it works differently than the real numbers in other ways too.

We can have AB = AC, but have B different than C with matrices (not true for real numbers).   (See Example 3)

We can have AD = 0, but A and D are not 0. (See Example 3)

The Identity Matrix:

A is m x n:

A * I =

I * A =

Theorem: A square (n x n) matrix is either row equivalent to I or row-reduces to something with a row of zeroes.

Proof: row reduce and get stuck or do not get stuck!

a square matrix A is said to be *invertible* or *nonsingular* if there is a square matrix B such that AB = BA = I

Ex. $A = \begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix}$ $B = \begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix}$

Theorem: Matrix inverses are unique.

Proof: Suppose B and C are both inverses of A.

$AB = I$
$C(AB) = CI$
$(CA) B = C$
$IB = C$
$B = C$

Inverse of 2x2 matrix: